Continuity of Maximal Monotone Operators in Banach Spaces(Nonlinear Analysis and Convex Analysis)

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バナッハ空間における極大単調作用素の連続性

(Continuity of Maximal Monotone Operators in Banach Spaces)

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1 Introduction

Let E be a real Banach space, E^* its dual space with pairing between x in E and x^* in E^* denoted by $\langle x, x^*\rangle$. Let T be an operator on E to 2^{E^*}. The domain of T is the set $D(T) = \{x \in E \mid Tx \neq \emptyset\}$. The graph of T is the subset of $E \times E^*$ given by $G(T) = \{(x, x^*) \in E \times E^* \mid x^* \in Tx\}$. The range of T is the set $R(T) = \bigcup\{Tx \mid x \in D(T)\}$. T is said to be monotone if for all $(x, x^*), (y, y^*) \in G(T)$,

$$\langle x - y, x^* - y^*\rangle \geq 0.$$

It is called maximal monotone if, in addition, its graph $G(T)$ is not properly contained in the graph of any other monotone operators on E. The notion of maximal monotone operators allows one to treat convex minimization problems as suitable problems associated with maximal monotone operators. In fact, let f be a proper lower semicontinuous convex function f from E to $(-\infty, \infty]$. Then we obtain

$$f(x_0) = \min_{x \in E} f(x) \Longleftrightarrow 0 \in \partial f(x_0)$$

where ∂f is the subdifferential of f defined by

$$\partial f(x) = \{x^* \in E^* \mid f(y) \geq \langle y - x, x^*\rangle + f(x), \quad \forall y \in E\}$$

for all $x \in E$; see Problem 4.1.2 of [4]. According to the result of Rockafellar, the subdifferential ∂f is maximal monotone; see Theorem 4.4.5 of [4].

In this paper, we treat some continuity property (in particular, closedness) of maximal monotone operators.

2 Closedness of m-accretive operators

A Banach space E is said to be smooth if $\lim_{t \to 0}(\|x + ty\| - \|x\|)/t$ exists for all $x, y \in U$ where $U = \{x \in E \mid \|x\| = 1\}$. For example, if E^* is strictly convex, then E is smooth; see Problem 4.3.1 of [3]. The duality mapping J from E into 2^{E^*} is defined by $J(x) = \{x^* \in E^* \mid \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$ for all $x \in E$. If E is smooth, then J is single valued; see Theorem 4.3.1 of [3]. An operator A on a smooth Banach space E into 2^E is said to be accretive if for all $(x_1, y_1), (x_2, y_2) \in G(A)$,

$$\langle y_1 - y_2, J(x_1 - x_2) \rangle \geq 0.$$

An accretive operator A is called m-accretive if $R(I + \lambda A) = E$ for all $\lambda > 0$. It is known that in Hilbert spaces, an m-accretive operator is equivalent to a maximal monotone operator; see Theorem 3.5.4 of [4].

An operator A is called demiclosed if the following condition is satisfied: if $x_n \in D(A)$ for $n = 1, 2, \ldots$, $x_n \to x$ and if there are $y_n \in Ax_n$ such that $y_n \rightharpoonup y$, then $y \in Ax$. It is noted that we use the symbol $\to$ and $\rightharpoonup$ to denote the convergence in the strong and the weak topology respectively. An operator A is called locally bounded at $x \in E$ if there is a neighborhood U of x such that $A(U)$ is bounded.

Kenmochi[1] proved the following.

Theorem (Kenmochi). *Let E be a real Banach space. Let E^* be uniformly convex. Let A be an accretive operator on E to 2^E with open domain. Suppose that A is demiclosed and Ax is closed and convex for all $x \in D(A)$ and locally bounded at every point of $D(A)$. Then A is m-accretive.*

3 Closedness of maximal monotone operators

An operator T on E to 2^{E^*} is called demiclosed if the following condition is satisfied: if $x_n \in D(T)$ for $n = 1, 2, \ldots$, $x_n \to x$ and if there are $x_n^* \in Tx_n$ such that $x_n^* \rightharpoonup x^*$, then $x^* \in Tx$. For example, demicontinuous operators are demiclosed in reflexive Banach spaces. In this section, we show some relation between demiclosedness and maximality for monotone operators.

For our main result, we need the following result which was essentially proved by Showalter (see Remark of [2]) and here present the proof for the sake of completeness.

Theorem (Showalter). *Let E be a Banach space and let T be a monotone operator on E to 2^{E^*}. Let x belong to the interior of $D(T)$ and $y \in D(T)$, and set $x_t = (1-t)x + ty$ for $t \in [0, 1)$. Then $\bigcup\{Tx_t \mid t \in [0, 1)\}$ is bounded in E^*.*

Proof. Let $z \in E$. We may show that $\{\langle z, x_t^* \rangle \mid t \in [0, 1),\ x_t^* \in Tx_t\}$ is bounded. Let $t \in [0, 1)$ and $x_t^* \in Tx_t$. Choose $y^* \in Ty$. Since T is monotone, we have

$$0 \leq \langle y - x_t, y^* - x_t^* \rangle = (1-t)\langle y - x, y^* - x^* \rangle.$$

So we obtain

$$\langle y - x, y^* \rangle \geq \langle y - x, x_t^* \rangle.$$

Since x belongs to the interior of $D(T)$, there exists some positive integer n_0 such that $T\left(x+\frac{1}{n_0}\right) \neq \emptyset$ and $T\left(x-\frac{1}{n_0}\right) \neq \emptyset$. Choose $z_1^* \in T\left(x+\frac{1}{n_0}\right)$. By monotonicity of T, we have

$$0 \leq \langle x+\frac{1}{n_0}z - x_t, z_1^* - x_t^*\rangle = \langle \frac{1}{n_0}z - t(y-x), z_1^* - x_t^*\rangle.$$

So we obtain

$$\langle \frac{1}{n_0}z - t(y-x), z_1^*\rangle + t\langle y-x, x_t^*\rangle \geq \frac{1}{n_0}\langle z, x_t^*\rangle.$$

Therefore we have

$$\begin{aligned}\langle z, x_t^*\rangle &\leq n_0\left(t\langle y-x, x_t^*\rangle + \frac{1}{n_0}\langle z, z_1^*\rangle - t\langle y-x, z_1^*\rangle\right)\\ &\leq n_0\left(t\langle y-x, y^*\rangle + \frac{1}{n_0}\langle z, z_1^*\rangle - t\langle y-x, z_1^*\rangle\right)\\ &\leq n_0\left(|\langle y-x, y^*\rangle| + \frac{1}{n_0}\langle z, z_1^*\rangle + |\langle y-x, z_1^*\rangle|\right).\end{aligned}$$

Next choose $z_2^* \in T\left(x-\frac{1}{n_0}\right)$. Since T is monotone, we also have

$$0 \leq \langle x-\frac{1}{n_0}z - x_t, z_2^* - x_t^*\rangle = \langle -\frac{1}{n_0}z - t(y-x), z_2^* - x_t^*\rangle.$$

It follows that

$$\langle -\frac{1}{n_0}z - t(y-x), z_2^*\rangle + t\langle y-x, x_t^*\rangle \geq -\frac{1}{n_0}\langle z, x_t^*\rangle.$$

So we have

$$\begin{aligned}-\langle z, x_t^*\rangle &\leq n_0\left(t\langle y-x, x_t^*\rangle - \frac{1}{n_0}\langle z, z_2^*\rangle - t\langle y-x, z_2^*\rangle\right)\\ &\leq n_0\left(t\langle y-x, y^*\rangle - \frac{1}{n_0}\langle z, z_2^*\rangle - t\langle y-x, z_2^*\rangle\right)\\ &\leq n_0\left(|\langle y-x, y^*\rangle| - \frac{1}{n_0}\langle z, z_2^*\rangle + |\langle y-x, z_2^*\rangle|\right).\end{aligned}$$

These show that $\langle z, x_t^*\rangle$ is bounded independently of $t \in [0,1)$ and $x_t^* \in Tx_t$. Hence we obtain that $\{\langle z, x_t^*\rangle \mid t \in [0,1),\ x_t^* \in Tx_t\}$ is bounded. The desired result follows by the uniform boundedness theorem. $\square$

Motivated by the result by Kenmochi, we obtain the following.

Theorem A. *Let E be a reflexive Banach space. Let T be an operator on E to 2^{E^*}. Suppose that $D(T) = E$. Then T is maximal monotone if and only if T is a monotone, demiclosed operator such that for all $x \in E$, Tx is a closed convex set.*

Proof. Assume first that T is maximal monotone. We show that T is a monotone, demiclosed operator such that for all $x \in E$, Tx is a closed convex set. As T is maximal monotone, we have

$$Tx = \bigcap_{(y,y^*)\in G(T)} \{x^* \in E^* \mid \langle x - y, x^* - y^* \rangle \geq 0\}$$

for all $x \in D(T)$. Then Tx is convex and weakstar closed because in the above intersection each set has these two properties. Since E is reflexive, we have Tx is a closed set.

Suppose that $x_n \in D(T)$ for $n = 1, 2, \ldots$, $x_n \to x$ and there are $x_n^* \in Tx_n$ such that $x_n^* \rightharpoonup x^*$. We show that $x^* \in Tx$. Let $(y, y^*) \in G(T)$. Since T is monotone, we have

$$\langle y - x_n, y^* - x_n^* \rangle \geq 0.$$

So we obtain

$$\begin{aligned}
\langle y - x, y^* - x^* \rangle &= \langle y - x, y^* - x_n^* \rangle + \langle y - x, x_n^* - x^* \rangle \\
&= \langle y - x_n, y^* - x_n^* \rangle + \langle x_n - x, y^* - x_n^* \rangle + \langle y - x, x_n^* - x^* \rangle \\
&\geq \langle x_n - x, y^* - x_n^* \rangle + \langle y - x, x_n^* - x^* \rangle \\
&\geq -\|x_n - x\| \|y^* - x_n^*\| + \langle y - x, x_n^* - x^* \rangle.
\end{aligned}$$

This yields

$$\langle y - x, y^* - x^* \rangle \geq 0.$$

Since T is maximal monotone, $(x, x^*) \in G(T)$.

We now assume that T is a monotone, demiclosed operator such that for all $x \in E$, Tx is a closed convex set. Then there exists a monotone operator S such that $G(T) \subset G(S)$ and $G(T) \neq G(S)$. Let (x_0, x_0^*) be a point in $G(S) \setminus G(T)$. Since $x_0^* \notin Tx_0$ and Tx_0 is weakly closed convex, there exists $z \in E$ satisfying

$$\langle z, x_0^* \rangle > \sup_{x^* \in Tx_0} \langle z, x^* \rangle. \tag{1}$$

We put

$$x_n = \left(1 - \frac{1}{n}\right) x_0 + \frac{1}{n} (x_0 + z), \quad n = 1, 2, \ldots.$$

By Showalter's Theorem, $\bigcup_{n=1}^{\infty} Tx_n$ is bounded. We choose $x_n^* \in Tx_n$ for each n. Since E is reflexive, there exists a subsequence $(x_{n_i}^*)$ such that $(x_{n_i}^*)$ converges weakly to some $p^* \in E^*$. Since T is demiclosed, we have $p^* \in Tx_0$.

Since $x_0^* \in Sx_0$, $x_{n_i}^* \in Sx_{n_i}$ and S is monotone,

$$\langle x_0 - x_{n_i}, x_0^* - x_{n_i}^* \rangle \geq 0,$$

so that

$$\langle z, x_0^* - x_{n_i}^* \rangle \leq 0.$$

We see from this that

$$\langle z, x_0^* - p^* \rangle \leq 0.$$

So we obtain

$$\langle z, x_0^* \rangle \leq \langle z, p^* \rangle. \tag{2}$$

From (1) and (2), we see that

$$\langle z, x_0^* \rangle > \sup_{x^* \in Tx_0} \langle z, x^* \rangle \geq \langle z, p^* \rangle \geq \langle z, x_0^* \rangle.$$

Hence we deduce a contradiction. □

Remark 1. Let E be a reflexive Banach space. It is known that the duality mapping J is maximal monotone. We prove it using Theorem A. First, J is monotone and Jx is a nonempty closed convex set for $x \in E$; see Theorem 4.2.1 of [3]. Furthermore J is demiclosed. Let $x_n \to x$, $x_n^* \in Jx_n$ and $x_n^* \rightharpoonup x^*$. We may show $x^* \in Jx$. Since the norm of E^* is lower semicontinuous in the wek topology, we have

$$\|x^*\| \leq \liminf_{n\to\infty} \|x_n^*\| = \liminf_{n\to\infty} \|x_n\| = \|x\|.$$

On the other hand, we have

$$|\langle x, x^* \rangle - \|x_n\|^2| = |\langle x, x^* \rangle - \langle x_n, x_n^* \rangle| \leq |\langle x, x^* - x_n^* \rangle| + |\langle x - x_n, x_n^* \rangle|.$$

So we obtain that $|\langle x, x^* \rangle - \|x_n\|^2| \to 0$. Then we have $\langle x, x^* \rangle = \|x\|^2$. Therefore $\|x\|^2 = \langle x, x^* \rangle \leq \|x\|\|x^*\|$, and we have $\|x\| \leq \|x^*\|$. These imply that $\langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2$ and hence $x^* \in Jx$. By Theorem A, J is maximal monotone.

Remark 2. Main result of this paper is related to our preceding work[5]. Let T be an operator on E to 2^{E^*}. We define T psuedo-continuous at $x \in D(T)$ if for each weakstar open set V with $V \subset Tx$, there exists a neighborhood U of x such that $z \in U$ implies $Tz \cap V \neq \emptyset$. Using this continuity, we obtain the following.

Theorem ([5]). *Let E be a real Banach space. Let T be an operator on E to 2^{E^*}. Suppose that $D(T) = E$. Then T is maximal monotone if and only if T is a monotone, psuedo-continuous operator such that for all $x \in E$, Tx is a weakstar closed convex set.*

References

[1] N. Kenmochi, *Accretive mappings in Banach spaces*, Hiroshima Math. J. **2** (1972), 163–177.

[2] R. E. Showalter, *Continuity of maximal monotone sets in Banach space*, Proc, Amer. Math. Soc. **42** (1973), 543–546.

[3] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

[4] W. Takahashi, *Convex Analysis and Approximation of Fixed Points*, Yokohama Publishers, Yokohama, 2000. (in Japanese)

[5] M. Toyoda, *Continuity of set-valued monotone mappings*, Proc. Internat. Conf. Nonlinear Anal. Convex Anal. (Hirosaki, 2001), 501–505.